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## An iterative method to obtain the specimen-independent three-parameter Weibull distribution of strength from bending tests

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### Abstract

Brittle materials, such as glass and ceramics, usually present a large strength scatter. Among other probability distributions, the Weibull distribution is widely used to characterize their resistance. Often the two-parameter model is employed, omitting the consideration of a threshold stress, leading to a simplified estimation method. For the sake of generality the present work uses fracture data from bending tests to obtain a three-parameter Weibull distribution function valid for a uni-axially and uniformly tensioned material element. The variable stress state prevailing in the flexural specimen and the size effect are simultaneously accounted for by means of an iterative fitting procedure. The method is extended to account for bimodal flaw distributions, discriminating between edge and surface failure results based on experimental observation. Finally, the model is applied to simulated data sets, obtaining satisfactory results.

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### 1. Introduction

As brittle materials present a high strength scatter, their resistance is in general described by a distribution function and not only by a single percentile value. The Weibull cumulative distribution function (cdf) of fracture stress is often employed, however frequently omitting the consideration of a

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threshold stress. The Weibull cdf [1] was developed for a uni-axially and uniformly tensioned material element, however, bending tests are characterized by a varying stress state over the length of the beam. Furthermore the size effect, i.e. decreasing strength with augmenting stressed area or length, has to be considered. Taking into account these issues the authors developed a method to obtain the primary three-parameter Weibull cdf of fracture stress from uni-axial bending tests [2]. This method also permits the combination of test data from different-sized specimens or data obtained by different test arrangements (e.g. 3-point and 4-point bending), in order to estimate the “pooled” Weibull parameters. The present paper extends the method to the parameter estimation in presence of concurrent flaw populations, a phenomenon sometimes found in brittle materials [3, 4]. For example in glass, edge and surface defects are present at the same time and the test data is usually a mixture of both. It is important to note, that it is statistically incorrect to simply discard fractures from population A at the time of estimating the parameters for population B or vice versa, resulting in too conservative estimates. The simplest solution would consist in evaluating the fracture data without considering different failure sources, but in general one would like to know the resistance of the different flaw distributions separately, because the data is required to predict failure for structures with different relations between edge length and surface area. To this end, the technique developed by Johnson [5] is used to rank correctly the separated failure data keeping in mind the influence of both failure mechanisms.

The whole sample can be described by a joint cdf assuming that the failure populations are independent. The proposed method is applied to simulated fracture data of 3-point bending tests, thus allowing us a direct comparison of the theoretical and estimated cdfs for each population.

## 2. Probabilistic model and proposed methodology

### 2.1. Probabilistic background

Under the assumption, that the cdf describing the failure probability of a uni-axially tensioned area  $\Delta A$  can be expressed in terms of the uni-axially acting stress  $\sigma$  in the form of a three-parameter Weibull distribution function  $F_{\Delta A}(\sigma)$  [1], the failure probability  $P_{f,Aref}$  for a uni-axially tensioned area  $A_{ref}$  with different dimensions than  $\Delta A$ , is given by

$$P_{f,Aref}(\sigma) = 1 - \exp \left[ -\frac{A_{ref}}{\Delta A} \left( \frac{\sigma - \lambda}{\delta} \right)^\beta \right], \quad (1)$$

$\lambda$  being the threshold stress,  $\delta$  the scale parameter referred to  $\Delta A$  and  $\beta$  the shape parameter. In a 4-point bending test the reference area  $A_{ref}$  depends not only on the dimensions of the test specimen, but also on the Weibull parameters and the fracture stress:

$$A_{ref} = w \cdot \left[ \frac{2 \cdot L_0}{(\beta + 1)} \cdot \left( 1 - \frac{\lambda}{\sigma} \right) + L_1 \right], \quad (2)$$

wherein the symbols refer to Fig. 1 and  $\sigma$  is the maximum tensile stress on the lower surface in the centre of the beam. Thus, the raw test data do not follow a “pure” three-parameter Weibull cdf. For 3-point bending  $L_1$  is equal to 0.

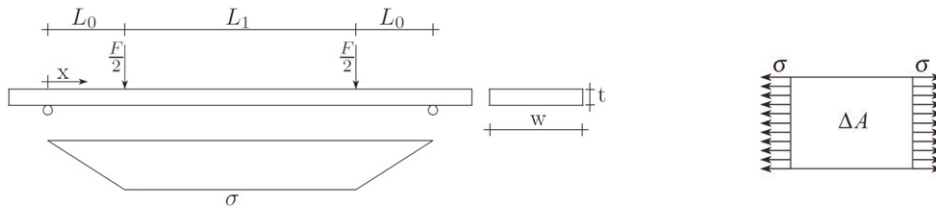


Fig. 1. a) 4-point bending test; b) uni-axially tensioned material element

The failure stresses obtained in an experiment of sample size  $N$  are sorted in ascending order and assigned to an accumulated failure probability given by

$$P_{f,i,beam} = \frac{i - 0.3}{N + 0.4}. \quad (3)$$

Considering statistical independence between the areas composing the beam, these failure probabilities are converted into failure probabilities referred to the elemental area  $\Delta A$  by

$$P_{f,i,\Delta A} = 1 - (1 - P_{f,i,beam})^{\Delta A / A_{ref,i}}. \quad (4)$$

These “shifted” data are fitted to a three-parameter Weibull cdf of fracture stress referred to a material element  $\Delta A$  using linear regression in a Weibull paper [6]. However, as  $A_{ref,i}$  depends on the unknown parameters  $\lambda$  and  $\beta$  (see Eq. (2)), an iteration process is carried out until the  $\beta$ - and  $\lambda$ -values used to calculate  $A_{ref,i}$  become equal to those estimated by linear regression of the data points referred to  $\Delta A$ . A complete deduction and detailed description of the method can be found in [2].

## 2.2. Competing failure modes

In case of concurrent flaw populations, the failure mode provoking fracture has to be known for each fracture stress. At the time of estimating the distribution function for failures due to surface defects, failures due to edge defects are considered as run-outs and vice versa. For this purpose, all fracture stresses are sorted in ascending order and a mean order number  $k_i$  is assigned to each failure stress  $\sigma_i$  caused by the considered failure population, using the equation proposed by Johnson [5], which is also employed by the CARES/LIFE program developed by the NASA [7] for the consideration of competing failure modes

$$k_i = k_{i-1} + \frac{N + 1 - k_{i-1}}{1 + R_i}, \quad (5)$$

being  $N$  the total sample size (including all failures from both populations) and  $R_i$  the remaining sample size (i.e. the number of specimens with a higher fracture stress than the specimen  $i$  including the specimen  $i$  itself). To calculate the failure probability, in Eq. (3)  $i$  is substituted by  $k_i$ . Each population is now fitted separately to a primary three-parameter Weibull cdf referred to a uni-axially and uniformly tensioned area  $\Delta A$  or length  $\Delta L$ , for surface or edge flaws, respectively. The iterative procedure is the same as explained in section 2.1., although the edge defects will be related to a reference length obtained by substituting the width  $w$  in Eq. (2) by 2, as there are two tensioned edges in the beam.

Once the cdf for each flaw population is determined and assuming independence between both, the joint cdf is expressed by

$$P_{f,joint}(\sigma) = \begin{cases} 1 - \exp \left[ -\frac{A_{ref}}{\Delta A} \left( \frac{\sigma - \lambda_A}{\delta_A} \right)^{\beta_A} - \frac{L_{ref}}{\Delta L} \left( \frac{\sigma - \lambda_L}{\delta_L} \right)^{\beta_L} \right], & \text{for } \sigma > \lambda_A \text{ and } \sigma > \lambda_L, \\ 1 - \exp \left[ -\frac{A_{ref}}{\Delta A} \left( \frac{\sigma - \lambda_A}{\delta_A} \right)^{\beta_A} \right], & \text{for } \sigma > \lambda_A \text{ and } \sigma < \lambda_L, \\ 1 - \exp \left[ -\frac{L_{ref}}{\Delta L} \left( \frac{\sigma - \lambda_L}{\delta_L} \right)^{\beta_L} \right], & \text{for } \sigma < \lambda_A \text{ and } \sigma > \lambda_L, \\ 0, & \text{for } \sigma < \lambda_A \text{ and } \sigma < \lambda_L. \end{cases} \quad (6)$$

Eq. (6) shows the three-parameter form of the multiplicative bimodal distribution often referred to in literature [4, 8], and is obtained by resting the product of the survival probabilities of both populations from 1. For a determined combination of two flaw populations, depending on the relation between the threshold stresses  $\lambda_A$  and  $\lambda_L$ , only three out of the four mentioned cases are valid.

To confirm the validity of the approach, all fracture stresses from both populations are sorted in ascending order and are assigned to the plotting positions, as if handling a unimodal distribution. Those data should be reasonably well represented by the joint distribution function given by Eq.(6), inserting the obtained parameter estimates for  $\lambda_A$ ,  $\delta_A$ ,  $\beta_A$ ,  $\lambda_L$ ,  $\delta_L$  and  $\beta_L$ .

### 3. Results for simulated failure data

The simulation of 3-point bending tests (length: 300 mm; width: 50 mm) and the subsequent parameter estimation have been carried out with Matlab<sup>®</sup>. For the theoretical defect populations the following parameters have been chosen for the three-parameter Weibull cdfs:

- surface defects:  $\lambda_A = 40$  MPa,  $\beta_A = 2.5$  and  $\delta_A = 90$  MPa referred to  $\Delta A = 225$  mm<sup>2</sup>;
- edge defects:  $\lambda_L = 35$  MPa,  $\beta_L = 2.0$  and  $\delta_L = 120$  MPa referred to  $\Delta L = 15$  mm.

To simulate the fracture data, two random numbers between 0 and 1 were generated representing a failure probability  $P_f$  for each population following the distributions

$$P_{f,Y}(\sigma) = 1 - \exp \left[ -\frac{Y_{ref}}{\Delta Y} \left( \frac{\sigma - \lambda_Y}{\delta_Y} \right)^{\beta_Y} \right], Y = A, L. \quad (7)$$

Solving for  $\sigma$ , the fracture stress is obtained for each population. The smaller value of the pair is registered along with the population where it originates from. This process is repeated until the sample size (N=100) is reached. Thus, the relation between the number of fractures due to surface and edge flaws is not constant, but varies for each simulation. The obtained fracture stresses are sorted in ascending order, separated and ranked as explained in section 2.2. Afterwards the two data subsets are fitted to three-parameter Weibull cdfs as pointed out in section 2.1. For the estimation of the Weibull-parameters the fitting proposed in [9] is used changing the search interval for  $\lambda$  to  $[\min(\sigma) - \delta^*[-1/n \cdot \log(0.01)]^{1/\beta}; \min(\sigma) - \delta^*[-1/n \cdot \log(0.99)]^{1/\beta}]$  according to [10]. Fig. 2 shows the cdfs for both populations separately and the joint cdf. For low failure probabilities the estimates are quite close to the theoretical curves, increasing the discrepancy at higher failure probabilities. This seems logical as for the primary cdfs the data points are located only in the left-hand tail allowing a good fit in this region. The over-all estimate for the joint cdf is however very close to the theoretical curve in all probability regions.

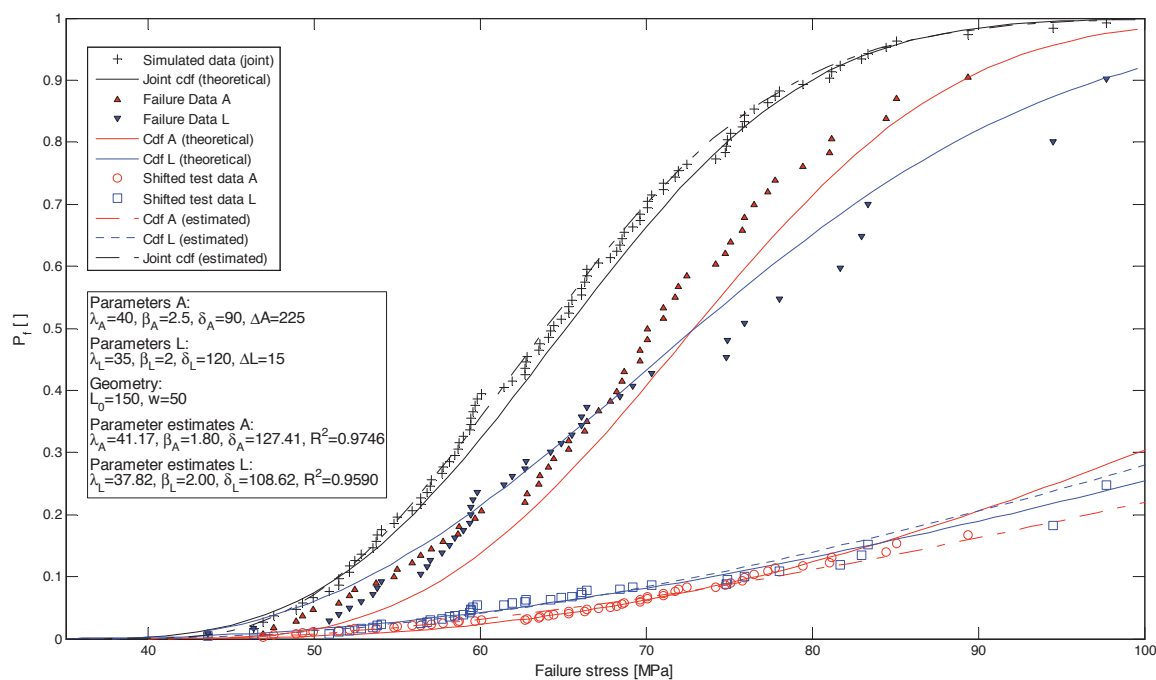


Fig. 2. Cumulative distribution functions (A: surface defects; L: edge defects)

To obtain an idea of the suitability of the method, 100 simulations, with different samples of failure stresses, have been carried out, and the percentile values compared to the theoretical values. Table 1 contains some characteristic values of these percentiles. The relative bias is calculated by  $(exact\ value - mean)/exact\ value$  and the safety coefficient is the ratio  $exact\ value/maximal\ value$ . In general, satisfying results are obtained, being the difference between the exact value and the mean less than 2 MPa and the relative bias less than 2 % for the calculated percentiles.

Table 1. Comparison of percentiles in 100 simulations

Percentiles	Surface defects			Edge defects		
	1 %	5 %	10 %	1 %	5 %	10 %
Exact value [MPa]	54.29	67.43	76.59	47.03	62.18	73.95
Mean [MPa]	53.62	67.14	77.99	46.92	61.92	74.64
Standard deviation [MPa]	1.20	1.72	3.09	1.77	1.82	4.16
Relative bias [%]	1.23%	0.43%	1.83%	0.23%	0.41%	0.93%
Maximal value [MPa]	56.87	71.28	91.31	52.30	66.92	97.54
Minimal value [MPa]	50.99	63.77	71.77	42.97	57.69	67.38
Safety coefficient [ ]	0.95	0.95	0.84	0.90	0.93	0.76

The increasing trend of the relative bias and the decreasing trend of the safety coefficient for higher percentiles indicate a higher accuracy for low failure probabilities.

Furthermore, the procedure proves that for a high number of data points ( $N=1000$ ) the estimates fit the theoretical curves.

#### 4. Conclusions

A method is proposed to estimate the three-parameter Weibull cdf of fracture stress considering simultaneously the size effect and varying stress state present in bending test specimens. The obtained cdf is related to a uni-axially and uniformly tensioned material element and can therefore easily be used in subsequent finite element analyses. The iterative method has been applied to simulated failure data of 3-point bending tests exhibiting surface and edge failure populations with satisfactory results. As the parameter estimates are mostly concerned with the left-hand tail of the cdf, extrapolation to higher failure probabilities than covered by the data points is not recommended.

Although the method is not recommended if only few fractures are assigned to the population, for which the parameter estimate is needed, this does not affect the quality of the fitting of the other population. Furthermore, it is possible to estimate the cdfs of more than two failure populations although this paper is handling only two populations, namely surface and edge flaws. The method can be used to characterize different cutting and edge finishing processes.

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